

# Composition Series and Jordan-Holder

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## Abstract

The Jordan-Holder Theorem states, more or less, that every finite group factors uniquely into simple groups (i.e. groups with no non-trivial proper normal subgroups). Furthermore, any infinite group that factors into simple groups must also factor uniquely.

## 1 Preliminaries

I assume you are familiar with

1. the axioms of group theory,
2. what a normal subgroup is, and that  $H, K \triangleleft G$  implies  $HK \triangleleft G$ ,
3. what a coset is,
4. what a quotient group is,
5. and the First, Second, and Fourth Isomorphism Theorems.

(This list is intentionally constructed in an order that makes it convenient to review these topics.)

As a reminder, here are the Second and Fourth Isomorphism Theorems.

### Theorem 1.1 (Second Isomorphism Theorem).

Let  $G$  be a group,  $H \leq G$ , and  $N \leq N_G(H)$ . Then  $NH/H \cong N/N \cap H$ , where all subgroups and quotients are well-defined.

### Theorem 1.2 (Fourth Isomorphism Theorem).

Let  $G$  be a group and  $N \trianglelefteq G$ . Then there is a bijection

$$f: \{H \mid N \leq H \leq G\} \rightarrow \{Q \mid Q \leq G/N\}$$

defined as  $f(H) = H/N$ . Furthermore,

1.  $H_1 \leq H_2 \iff H_1/N \leq H_2/N$
2.  $H \trianglelefteq G \iff H/N \trianglelefteq G/N$ .

So  $f$  preserves subgroups and normality of subgroups of  $G$ .

There are a garden variety of other properties that this bijection satisfies. None of them are important for our purposes here.

Now onto new content. We first start by defining a simple group and a composition series of a group.

**Definition 1.3 (Simple Groups).**

A group  $G$  is simple if it has exactly two normal subgroups:  $1$  and  $G$ .

Another way to think of this definition is that  $G$  has no interesting normal subgroups. **Note 1 is not a simple group as it only has one normal subgroup.**

**Definition 1.4 (Composition Series).**

A composition series of  $G$  is a **finite** sequence of subgroups

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

where  $G_{i+1}/G_i$  is simple for all  $0 \leq i < n$ .

We refer to  $G_{i+1}/G_i$  as a **composition factor** of  $G$ .

Here is an example: take  $Z_6$  (the cyclic group of order 6). The only two composition series are

$$1 \triangleleft Z_2 \triangleleft Z_6$$

and

$$1 \triangleleft Z_3 \triangleleft Z_6.$$

(What are the quotient groups of successive subgroups in each composition series?)

An important fact is that all finite groups have a composition series. The argument goes like this: suppose we know that all groups with order less than  $n$  have a composition series. Then take a group  $G$  with  $|G| = n$ . Either  $G$  is simple, in which case we are done ( $1 \triangleleft G$ ), or it has an interesting normal subgroup  $H$ .

In the latter case, take a composition series

$$1 = H_0 \triangleleft \cdots \triangleleft H_n = H$$

and a composition series

$$H/H = S_0/H \triangleleft \cdots \triangleleft S_m/H = G/H$$

But note that by the Fourth Isomorphism Theorem,

$$1 = H_0 \triangleleft \cdots \triangleleft H \triangleleft S_1 \triangleleft \cdots \triangleleft S_m = G$$

is a composition series of  $G$ .

This fact does not directly relate to the statement of the Jordan-Holder Theorem. Yet it illuminates why we care: composition series are ubiquitous in finite group theory.

## 2 Jordan-Holder

Informally, Jordan-Holder states that any two composition series have the same length, and that their composition factors are the same (up to isomorphism and rearrangement).

**Theorem 2.1 (Jordan-Holder).**

Suppose  $G$  has two composition series

$$1 = G_0 \triangleleft \cdots \triangleleft G_n = G$$

and

$$1 = H_0 \triangleleft \cdots \triangleleft H_m = H.$$

Then there is a bijection  $\pi: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$  such that

$$G_{\pi(i)}/G_{\pi(i-1)} = H_i/H_{i-1}$$

for all  $0 \leq i < n$ .

The existence of a bijection implies  $n = m$ .

**2.1 Outline****2.1.1 Strategy**

We will utilize strong induction.

Take the smallest  $k$  such that  $H_k$  is not a subgroup of  $G_{n-1}$ . Here is our strategy: we are going to consider the series

$$1 = H_0 \cap G_{n-1} \trianglelefteq \cdots \trianglelefteq H_m \cap G_{n-1} = G_{n-1}.$$

This isn't a composition series of  $G_{n-1}$  (in fact, it better not be, because it has a length of  $m$  while we know  $1 \triangleleft \cdots \triangleleft G_{n-1}$  has length  $n-1$ ). But we will show that removing  $H_k \cap G_{n-1}$  makes it one.

**2.1.2 The repeat**

We make the crucial observation that  $H_k \cap G_{n-1} = H_{k-1}$ . This is because  $H_{k-1}$  and  $H_k \cap G_{n-1}$  are both normal subgroups of  $H_k$ , so  $H_{k-1}(H_k \cap G_{n-1}) \triangleleft H_k$  too. Thus

$$\frac{H_{k-1}(H_k \cap G_{n-1})}{H_{k-1}} \triangleleft \frac{H_k}{H_{k-1}}.$$

Note  $\frac{H_k}{H_{k-1}}$  is simple and  $H_{k-1}(H_k \cap G_{n-1})$  is a subgroup of  $G_{n-1}$ , so it cannot be equal to  $H_k$ . So  $H_k \cap G_{n-1} = H_{k-1}$ .

**2.1.3 Using inductions to establish most isomorphisms**

Now we are going to consider the composition series

$$(H_0 \cap G_{n-1}) \triangleleft \cdots \triangleleft (H_{k-1} \cap G_{n-1}) \triangleleft (H_{k+1} \cap G_{n-1}) \triangleleft \cdots \triangleleft (H_m \cap G_{n-1})$$

of  $G_{n-1}$  (note that we have removed  $H_k \cap G_{n-1}$ ) which has composition factors

$$\frac{H_1 \cap G_{n-1}}{H_0 \cap G_{n-1}}, \dots, \frac{H_{k-1} \cap G_{n-1}}{H_{k-2} \cap G_{n-1}}, \frac{H_{k+1} \cap G_{n-1}}{H_k \cap G_{n-1}}, \dots, \frac{H_m \cap G_{n-1}}{H_{m-1} \cap G_{n-1}}.$$

Then we show it is isomorphic to the sequence of composition factors

$$\frac{H_1}{H_0}, \dots, \frac{H_{k-1}}{H_{k-2}}, \frac{H_{k+1}}{H_k}, \dots, \frac{H_m}{H_{m-1}}.$$

But also, as  $H_0 \cap G_{n-1} \triangleleft \cdots \triangleleft H_m \cap G_{n-1}$  is a composition series of  $G_{n-1}$ , its composition factors must also be isomorphic to the composition factors of

$$G_0 \triangleleft \cdots \triangleleft G_{n-1}.$$

### 2.1.4 The Final Correspondence

So we have set up a correspondence between most of the composition factors of  $G_0 \triangleleft \cdots \triangleleft G_n$  and  $H_0 \triangleleft \cdots \triangleleft H_m$ . One pair remains: we need to show

$$\frac{G}{G_{n-1}} \cong \frac{H_k}{H_{k-1}}.$$

This is true as

$$\frac{G}{G_{n-1}} = \frac{H_k G_{n-1}}{G_{n-1}} \cong \frac{H_k}{H_k \cap G_{n-1}} = \frac{H_k}{H_{k-1}}.$$

## 2.2 Proof

We utilize strong induction on  $\max(n, m)$ .

Take the smallest  $k$  such that  $H_k$  is not a subgroup of  $G_{n-1}$ .

**Lemma 2.2.**  $H_k \cap G_{n-1} = H_{k-1}$ .

**Proof.** Note that  $H_{k-1} \triangleleft H_k$  by definition and  $H_k \cap G_{n-1} \triangleleft H_k$  as  $G_{n-1} \triangleleft G$ .<sup>1</sup>

Obviously  $H_{k-1} \subseteq H_{k-1}(H_k \cap G_{n-1})$ . Combining this with the well-known fact that  $H, K \triangleleft G$  implies  $HK \triangleleft G$ , we note that

$$H_{k-1} \triangleleft H_{k-1}(H_k \cap G_{n-1}) \triangleleft H_k.$$
<sup>2</sup>

But  $\frac{H_k}{H_{k-1}}$  is simple, so by the Fourth Isomorphism Theorem, we must either have

$$H_{k-1} = H_{k-1}(H_k \cap G_{n-1}) \text{ or } H_k = H_{k-1}(H_k \cap G_{n-1}).$$

The latter is impossible since  $G_{n-1} \leq H_{k-1}(H_k \cap G_{n-1})$  yet  $G_{n-1} \not\leq H_k$ . So we have

$$H_{k-1} = H_{k-1}(H_k \cap G_{n-1}).$$

This implies  $H_k \cap G_{n-1} \leq H_{k-1}$ . But as  $H_{k-1} \leq H_k$  and  $H_{k-1} \leq G_{n-1}$ , we also have  $H_{k-1} \leq H_k \cap G_{n-1}$ . This establishes a double inclusion, so  $H_k \cap G_{n-1} = H_{k-1}$ , as desired.  $\square$

**Lemma 2.3.** Consider the series

$$\frac{H_0 G_{n-1}}{G_{n-1}}, \dots, \frac{H_m G_{n-1}}{G_{n-1}}.$$

If  $0 \leq i < k$  then

$$\frac{H_i G_{n-1}}{G_{n-1}} \cong 1,$$

and if  $k \leq i \leq m$  then

$$\frac{H_i G_{n-1}}{G_{n-1}} = \frac{G}{G_{n-1}}.$$

**Proof.** Let  $A$  and  $B$  be subgroups of  $G$ , and  $N$  be a normal subgroup of  $G$ . Recall that  $A \triangleleft B$  implies  $\frac{AN}{N} \triangleleft \frac{BN}{N}$ .<sup>3</sup> So

$$H_0 \triangleleft \cdots \triangleleft H_m$$

implies

$$1 \cong \frac{H_0 G_{n-1}}{G_{n-1}} \triangleleft \cdots \triangleleft \frac{H_m G_{n-1}}{G_{n-1}} = \frac{G}{G_{n-1}}.$$

<sup>1</sup>This relies on the following fact:  $H \leq G$  and  $N \triangleleft G$  implies  $H \cap N \triangleleft H$ . Proof: suppose  $n \in H \cap N$ . Then for all  $h \in H$ ,  $hnh^{-1} \in N$  as  $N$  is normal, and  $hnh^{-1} \in H$  as  $n \in H$ . So elements in  $H \cap N$  are closed under conjugation by elements in  $H$ , as desired.

<sup>2</sup>Since  $H_{k-1}$  is a normal subgroup of  $H_k$ , it must be a normal subgroup of any subgroup of  $H_k$  containing  $H_{k-1}$ . Since  $H_k \cap G_{n-1}$  is such a subgroup, we must have  $H_{k-1} \triangleleft H_k \cap G_{n-1}$ .

<sup>3</sup>Proof:  $(bN)(aN)(b^{-1}N) = bab^{-1}N$ , and we know  $bab^{-1} \in A$  so  $bab^{-1}N \in AN$ .

By the simplicity of  $\frac{G}{G_{n-1}}$ , each  $\frac{H_i G_{n-1}}{G_{n-1}}$  is either isomorphic to 1 or equivalent to  $\frac{G_n}{G_{n-1}}$ . If  $H_i \leq G_{n-1}$  the former is true. If  $H_i \not\leq G_{n-1}$  then the former cannot be true, so the latter is true.  $\square$

**Lemma 2.4.** For all  $i < k - 1$ ,

$$\frac{H_{i+1} \cap G_{n-1}}{H_i \cap G_{n-1}} \cong \frac{H_{i+1}}{H_i}.$$

**Proof.** This is obvious as  $H_i, H_{i+1} \leq G_{n-1}$ .  $\square$

**Lemma 2.5.** For all  $i \geq k$ ,

$$\frac{H_{i+1} \cap G_{n-1}}{H_i \cap G_{n-1}} \cong \frac{H_{i+1}}{H_i}.$$

**Proof.** We claim  $H_i(H_{i+1} \cap G_{n-1}) = H_{i+1}$ . It is a normal subgroup of  $H_{i+1}$  as  $H_i \triangleleft H_{i+1}$  and  $H_{i+1} \cap G_{n-1} \triangleleft H_{i+1}$ . Furthermore  $H_i$  is a normal subgroup of it.

By the Second Isomorphism Theorem and Lemma 2.3,

$$\frac{G}{G_{n-1}} = \frac{H_{i+1} G_{n-1}}{G_{n-1}} \cong \frac{H_{i+1}}{H_{i+1} \cap G_{n-1}}.$$

Note this quotient is simple.

Since

$$H_{i+1} \cap G_{n-1} \triangleleft H_i(H_{i+1} \cap G_{n-1}) \triangleleft H_{i+1},^4$$

simplicity of the quotient yields

$$H_i(H_{i+1} \cap G_{n-1}) = H_{i+1} \cap G_{n-1} \text{ or } H_i(H_{i+1} \cap G_{n-1}) = H_{i+1}.$$

Obviously the first case isn't true.<sup>5</sup> So it must be the second.

Now by the Second Isomorphism Theorem,

$$\frac{H_{i+1}}{H_i} = \frac{H_i(H_{i+1} \cap G_{n-1})}{H_i} \cong \frac{H_{i+1} \cap G_{n-1}}{H_i \cap (H_{i+1} \cap G_{n-1})} = \frac{H_{i+1} \cap G_{n-1}}{H_i \cap G_{n-1}},$$

as desired. (We have  $H_i \leq N_G(H_{i+1} \cap G_{n-1})$  as  $H_i \triangleleft H_{i+1}$  and  $H_{i+1} \cap G_{n-1} \leq H_{i+1}$ .)  $\square$

Combining our lemmas, we now turn to the series

$$(H_0 \cap G_{n-1}) \triangleleft \cdots \triangleleft (H_{k-1} \cap G_{n-1}) \triangleleft (H_{k+1} \cap G_{n-1}) \triangleleft \cdots \triangleleft (H_m \cap G_{n-1}).$$

Note its composition factors are isomorphic to

$$\frac{H_1}{H_0}, \dots, \frac{H_{k-1}}{H_{k-2}}, \frac{H_{k+1}}{H_k}, \dots, \frac{H_m}{H_{m-1}}.$$

Because each of these quotients are simple, the series is a composition series.

Now we resolve one potential hiccup: note

$$\frac{H_{k+1} \cap G_{n-1}}{H_{k-1} \cap G_{n-1}} = \frac{H_{k+1} \cap G_{n-1}}{H_{k-1}} = \frac{H_{k+1} \cap G_{n-1}}{H_k \cap G_{n-1}}.$$

So the  $H_{k-1} \cap G_{n-1}$  term is performing double duty.

As

$$G_0 \triangleleft \cdots \triangleleft G_{n-1}$$

is a composition series of  $G_{n-1}$ , by the inductive hypothesis, there is some bijection

$$\pi_0: \{1, \dots, n-1\} \rightarrow \{1, \dots, k-1, k+1, \dots, m-1\}$$

<sup>4</sup>Note  $H_{i+1} \cap G_{n-1} \triangleleft H_{i+1}$  so  $H_{i+1} \cap G_{n-1} \triangleleft H_i(H_{i+1} \cap G_{n-1})$ .

<sup>5</sup>Proof:  $G_{n-1} \not\leq H_i(H_{i+1} \cap G_{n-1})$  but  $G_{n-1} \leq H_{i+1} \cap G_{n-1}$ .

such that

$$\frac{G_{\pi_0(i)}}{G_{\pi_0(i-1)}} \cong \frac{H_i}{H_{i-1}}$$

for all  $1 \leq i \leq n-1$ .

**Lemma 2.6.** We have

$$\frac{G}{G_{n-1}} \cong \frac{H_k}{H_{k-1}}.$$

**Proof.** Note  $G = G_{n-1}H_k$  by Lemma 2.3 as  $G_{n-1}H_k \neq G_{n-1}$ . So

$$\frac{G}{G_{n-1}} = \frac{H_k G_{n-1}}{G_{n-1}},$$

which is isomorphic to

$$\frac{H_k}{H_k \cap G_{n-1}} = \frac{H_k}{H_{k-1}}$$

by the Second Isomorphism Theorem. (The equivalence is a consequence of Lemma 2.2.) □

Now we extend  $\pi_0$  to a bijection

$$\pi: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$$

where

$$\frac{G_{\pi(i)}}{G_{\pi(i-1)}} \cong \frac{H_i}{H_{i-1}}$$

for all  $1 \leq i \leq n$ . Easy:

$$\pi(i) = \begin{cases} \pi_0(i) & 1 \leq i \leq n-1, \\ k & i = n. \end{cases}$$

### 3 Does the converse hold?

The factorization of a group  $G$  into simple subgroups is unique, much as the factorization of a natural number  $n$  into primes is unique. Furthermore, the prime factors of a natural number uniquely determine said number.

This gives rise to a natural question: do the composition factors of a group  $G$  uniquely determine the group? In other words, does the converse of Jordan-Holder hold? The answer is no:

$$1 \triangleleft Z_p \triangleleft Z_{p^2}$$

and

$$1 \triangleleft Z_p \triangleleft Z_p \times Z_p.$$

The composition factors are identical (the only non-trivial check is  $Z_{p^2}/Z_p \cong Z_p \cong (Z_p \times Z_p)/Z_p$ ) yet  $Z_{p^2}$  and  $Z_p \times Z_p$  are not isomorphic.