Composition Series and Jordan-Holder

Dennis Chen

May 11, 2024

Abstract

The Jordan-Holder Theorem states, more or less, that every finite group factors uniquely into simple groups (i.e. groups with no non-trivial proper normal subgroups). Furthermore, any infinite group that factors into simple groups must also factor uniquely.

1 Preliminaries

I assume you are familiar with

1. the axioms of group theory,
2. what a normal subgroup is, and that $H, K \lhd G$ implies $HK \lhd G$,
3. what a coset is,
4. what a quotient group is,
5. and the First, Second, and Fourth Isomorphism Theorems.

(This list is intentionally constructed in an order that makes it convenient to review these topics.)

As a reminder, here are the Second and Fourth Isomorphism Theorems.

Theorem 1.1 (Second Isomorphism Theorem).

Let G be a group, $H \leq G$, and $N \leq N_G(H)$. Then $NH/H \cong N/N \cap H$, where all subgroups and quotients are well-defined.

Theorem 1.2 (Fourth Isomorphism Theorem).

Let G be a group and $N \trianglelefteq G$. Then there is a bijection

$$f \colon \{H \mid N \leq H \leq G\} \to \{Q \mid Q \leq G/N\}$$

defined as $f(H) = H/N$. Furthermore,

1. $H_1 \leq H_2 \iff H_1/N \leq H_2/N$
2. $H \trianglelefteq G \iff H/N \trianglelefteq G/N$.

So f preserves subgroups and normality of subgroups of G.

There are a garden variety of other properties that this bijection satisfies. None of them are important for our purposes here.

Now onto new content. We first start by defining a simple group and a composition series of a group.

Definition 1.3 (Simple Groups).

A group G is simple if it has exactly two normal subgroups: 1 and G.

Another way to think of this definition is that G has no interesting normal subgroups. **Note 1 is not a simple group as it only has one normal subgroup.**

Definition 1.4 (Composition Series).

A composition series of G is a **finite** sequence of subgroups

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

where G_{i+1}/G_i is simple for all $0 \leq i < n$.

We refer to G_{i+1}/G_i as a **composition factor** of G.

Here is an example: take Z_6 (the cyclic group of order 6). The only two composition series are

$$1 \lhd Z_2 \lhd Z_6$$

and

$$1 \lhd Z_3 \lhd Z_6.$$

(What are the quotient groups of successive subgroups in each composition series?)

An important fact is that all finite groups have a composition series. The argument goes like this: suppose we know that all groups with order less than n have a composition series. Then take a group G with $|G| = n$. Either G is simple, in which case we are done ($1 \lhd G$), or it has an interesting normal subgroup H.

In the latter case, take a composition series

$$1 = H_0 \lhd \cdots \lhd H_n = H$$

and a composition series

$$H/H = S_0/H \lhd \cdots \lhd S_m/H = G/H$$

But note that by the Fourth Isomorphism Theorem,

$$1 = H_0 \lhd \cdots \lhd H \lhd S_1 \lhd \cdots \lhd S_m = G$$

is a composition series of G.

This fact does not directly relate to the statement of the Jordan-Holder Theorem. Yet it illuminates why we care: composition series are ubiquitous in finite group theory.

2 Jordan-Holder

Informally, Jordan-Holder states that any two composition series have the same length, and that their composition factors are the same (up to isomorphism and rearrangement).

Theorem 2.1 (Jordan-Holder).

Suppose G has two composition series

$$1 = G_0 \lhd \cdots \lhd G_n = G$$

and

$$1 = H_0 \lhd \cdots \lhd H_m = H.$$

Then there is a bijection $\pi \colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that

$$G_{\pi(i)}/G_{\pi(i)-1} = H_i/H_{i-1}$$

for all $0 \le i < n$.

The existence of a bijection implies $n = m$.

2.1 Outline

2.1.1 Strategy

We will utilize strong induction.

Take the smallest k such that H_k is not a subgroup of G_{n-1}. Here is our strategy: we are going to consider the series

$$1 = H_0 \cap G_{n-1} \trianglelefteq \cdots \trianglelefteq H_m \cap G_{n-1} = G_{n-1}.$$

This isn't a composition series of G_{n-1} (in fact, it better not be, because it has a length of m while we know $1 \lhd \cdots \lhd G_{n-1}$ has length $n-1$). But we will show that removing $H_k \cap G_{n-1}$ makes it one.

2.1.2 The repeat

We make the crucial observation that $H_k \cap G_{n-1} = H_{k-1}$. This is because H_{k-1} and $H_k \cap G_{n-1}$ are both normal subgroups of H_k, so $H_{k-1}(H_k \cap G_{n-1}) \lhd H_k$ too. Thus

$$\frac{H_{k-1}(H_k \cap G_{n-1})}{H_{k-1}} \lhd \frac{H_k}{H_{k-1}}.$$

Note $\frac{H_k}{H_{k-1}}$ is simple and $H_{k-1}(H_k \cap G_{n-1})$ is a subgroup of G_{n-1}, so it cannot be equal to H_k. So $H_k \cap G_{n-1} = H_{k-1}$.

2.1.3 Using inductions to establish most isomorphisms

Now we are going to consider the composition series

$$(H_0 \cap G_{n-1}) \lhd \cdots \lhd (H_{k-1} \cap G_{n-1}) \lhd (H_{k+1} \cap G_{n-1}) \lhd \cdots \lhd (H_m \cap G_{n-1})$$

of G_{n-1} (note that we have removed $H_k \cap G_{n-1}$) which has composition factors

$$\frac{H_1 \cap G_{n-1}}{H_0 \cap G_{n-1}}, \ldots, \frac{H_{k-1} \cap G_{n-1}}{H_{k-2} \cap G_{n-1}}, \frac{H_{k+1} \cap G_{n-1}}{H_k \cap G_{n-1}}, \ldots, \frac{H_m \cap G_{n-1}}{H_{m-1} \cap G_{n-1}}.$$

Then we show it is isomorphic to the sequence of composition factors

$$\frac{H_1}{H_0}, \ldots, \frac{H_{k-1}}{H_{k-2}}, \frac{H_{k+1}}{H_k}, \ldots, \frac{H_m}{H_{m-1}}.$$

But also, as $H_0 \cap G_{n-1} \lhd \cdots \lhd H_m \cap G_{n-1}$ is a composition series of G_{n-1}, its composition factors must also be isomorphic to the composition factors of

$$G_0 \lhd \cdots \lhd G_{n-1}.$$

2.1.4 The Final Correspondence

So we have set up a correspondence between most of the composition factors of $G_0 \triangleleft \cdots \triangleleft G_n$ and $H_0 \triangleleft \cdots \triangleleft H_m$. One pair remains: we need to show

$$\frac{G}{G_{n-1}} \cong \frac{H_k}{H_{k-1}}.$$

This is true as

$$\frac{G}{G_{n-1}} = \frac{H_k G_{n-1}}{G_{n-1}} \cong \frac{H_k}{H_k \cap G_{n-1}} = \frac{H_k}{H_{k-1}}.$$

2.2 Proof

We utilize strong induction on $\max(n, m)$.

Take the smallest k such that H_k is not a subgroup of G_{n-1}.

Lemma 2.2. $H_k \cap G_{n-1} = H_{k-1}$.

Proof. Note that $H_{k-1} \triangleleft H_k$ by definition and $H_k \cap G_{n-1} \triangleleft H_k$ as $G_{n-1} \triangleleft G$.[1]

Obviously $H_{k-1} \subseteq H_{k-1}(H_k \cap G_{n-1})$. Combining this with the well-known fact that $H, K \triangleleft G$ implies $HK \triangleleft G$, we note that

$$H_{k-1} \triangleleft H_{k-1}(H_k \cap G_{n-1}) \triangleleft H_k.^{2}$$

But $\frac{H_k}{H_{k-1}}$ is simple, so by the Fourth Isomorphism Theorem, we must either have

$$H_{k-1} = H_{k-1}(H_k \cap G_{n-1}) \text{ or } H_k = H_{k-1}(H_k \cap G_{n-1}).$$

The latter is impossible since $G_{n-1} \leq H_{k-1}(H_k \cap G_{n-1})$ yet $G_{n-1} \not\leq H_k$. So we have

$$H_{k-1} = H_{k-1}(H_k \cap G_{n-1}).$$

This implies $H_k \cap G_{n-1} \leq H_{k-1}$. But as $H_{k-1} \leq H_k$ and $H_{k-1} \leq G_{n-1}$, we also have $H_{k-1} \leq H_k \cap G_{n-1}$. This establishes a double inclusion, so $H_k \cap G_{n-1} = H_{k-1}$, as desired. ☐

Lemma 2.3. Consider the series

$$\frac{H_0 G_{n-1}}{G_{n-1}}, \ldots, \frac{H_m G_{n-1}}{G_{n-1}}.$$

If $0 \leq i < k$ then

$$\frac{H_i G_{n-1}}{G_{n-1}} \cong 1,$$

and if $k \leq i \leq m$ then

$$\frac{H_i G_{n-1}}{G_{n-1}} = \frac{G}{G_{n-1}}.$$

Proof. Let A and B be subgroups of G, and N be a normal subgroup of G. Recall that $A \triangleleft B$ implies $\frac{AN}{N} \triangleleft \frac{BN}{N}$.[3] So

$$H_0 \triangleleft \cdots \triangleleft H_m$$

implies

$$1 \cong \frac{H_0 G_{n-1}}{G_{n-1}} \triangleleft \cdots \triangleleft \frac{H_m G_{n-1}}{G_{n-1}} = \frac{G}{G_{n-1}}.$$

[1]This relies on the following fact: $H \leq G$ and $N \triangleleft G$ implies $H \cap N \triangleleft H$. Proof: suppose $n \in H \cap N$. Then for all $h \in H$, $hnh^{-1} \in N$ as N is normal, and $hnh^{-1} \in H$ as $n \in H$. So elements in $H \cap N$ are closed under conjugation by elements in H, as desired.

[2]Since H_{k-1} is a normal subgroup of H_k, it must be a normal subgroup of any subgroup of H_k containing H_{k-1}. Since $H_k \cap G_{n-1}$ is such a subgroup, we must have $H_{k-1} \triangleleft H_k \cap G_{n-1}$.

[3]Proof: $(bN)(aN)(b^{-1}N) = bab^{-1}N$, and we know $bab^{-1} \in A$ so $bab^{-1}N \in AN$.

By the simplicity of $\frac{G}{G_{n-1}}$, each $\frac{H_iG_{n-1}}{G_{n-1}}$ is either isomorphic to 1 or equivalent to $\frac{G_n}{G_{n-1}}$. If $H_i \le G_{n-1}$ the former is true. If $H_i \not\le G_{n-1}$ then the former cannot be true, so the latter is true. □

Lemma 2.4. For all $i < k-1$,

$$\frac{H_{i+1} \cap G_{n-1}}{H_i \cap G_{n-1}} \cong \frac{H_{i+1}}{H_i}.$$

Proof. This is obvious as $H_i, H_{i+1} \le G_{n-1}$. □

Lemma 2.5. For all $i \ge k$,

$$\frac{H_{i+1} \cap G_{n-1}}{H_i \cap G_{n-1}} \cong \frac{H_{i+1}}{H_i}.$$

Proof. We claim $H_i(H_{i+1} \cap G_{n-1}) = H_{i+1}$. It is a normal subgroup of H_{i+1} as $H_i \lhd H_{i+1}$ and $H_{i+1} \cap G_{n-1} \lhd H_{i+1}$. Furthermore H_i is a normal subgroup of it.

By the Second Isomorphism Theorem and Lemma 2.3,

$$\frac{G}{G_{n-1}} = \frac{H_{i+1}G_{n-1}}{G_{n-1}} \cong \frac{H_{i+1}}{H_{i+1} \cap G_{n-1}}.$$

Note this quotient is simple.

Since

$$H_{i+1} \cap G_{n-1} \lhd H_i(H_{i+1} \cap G_{n-1}) \lhd H_{i+1},\text{[4]}$$

simplicity of the quotient yields

$$H_i(H_{i+1} \cap G_{n-1}) = H_{i+1} \cap G_{n-1} \text{ or } H_i(H_{i+1} \cap G_{n-1}) = H_{i+1}.$$

Obviously the first case isn't true.[5] So it must be the second.

Now by the Second Isomorphism Theorem,

$$\frac{H_{i+1}}{H_i} = \frac{H_i(H_{i+1} \cap G_{n-1})}{H_i} \cong \frac{H_{i+1} \cap G_{n-1}}{H_i \cap (H_{i+1} \cap G_{n-1})} = \frac{H_{i+1} \cap G_{n-1}}{H_i \cap G_{n-1}},$$

as desired. (We have $H_i \le N_G(H_{i+1} \cap G_{n-1})$ as $H_i \lhd H_{i+1}$ and $H_{i+1} \cap G_{n-1} \le H_{i+1}$.) □

Combining our lemmas, we now turn to the series

$$(H_0 \cap G_{n-1}) \lhd \cdots \lhd (H_{k-1} \cap G_{n-1}) \lhd (H_{k+1} \cap G_{n-1}) \lhd \cdots \lhd (H_m \cap G_{n-1}).$$

Note its composition factors are isomorphic to

$$\frac{H_1}{H_0}, \ldots, \frac{H_{k-1}}{H_{k-2}}, \frac{H_{k+1}}{H_k}, \ldots, \frac{H_m}{H_{m-1}}.$$

Because each of these quotients are simple, the series is a composition series.

Now we resolve one potential hiccup: note

$$\frac{H_{k+1} \cap G_{n-1}}{H_{k-1} \cap G_{n-1}} = \frac{H_{k+1} \cap G_{n-1}}{H_{k-1}} = \frac{H_{k+1} \cap G_{n-1}}{H_k \cap G_{n-1}}.$$

So the $H_{k-1} \cap G_{n-1}$ term is performing double duty.

As

$$G_0 \lhd \cdots \lhd G_{n-1}$$

is a composition series of G_{n-1}, by the inductive hypothesis, there is some bijection

$$\pi_0 \colon \{1, \ldots, n-1\} \to \{1, \ldots, k-1, k+1, \ldots, m-1\}$$

[4]Note $H_{i+1} \cap G_{n-1} \lhd H_{i+1}$ so $H_{i+1} \cap G_{n-1} \lhd H_i(H_{i+1} \cap G_{n-1})$.

[5]Proof: $G_{n-1} \not\le H_i(H_{i+1} \cap G_{n-1})$ but $G_{n-1} \le H_{i+1} \cap G_{n-1}$.

such that

$$\frac{G_{\pi_0(i)}}{G_{\pi_0(i)-1}} \cong \frac{H_i}{H_{i-1}}$$

for all $1 \le i \le n-1$.

Lemma 2.6. We have

$$\frac{G}{G_{n-1}} \cong \frac{H_k}{H_{k-1}}.$$

Proof. Note $G = G_{n-1}H_k$ by Lemma 2.3 as $G_{n-1}H_k \neq G_{n-1}$. So

$$\frac{G}{G_{n-1}} = \frac{H_k G_{n-1}}{G_{n-1}},$$

which is isomorphic to

$$\frac{H_k}{H_k \cap G_{n-1}} = \frac{H_k}{H_{k-1}}$$

by the Second Isomorphism Theorem. (The equivalence is a consequence of Lemma 2.2.) □

Now we extend π_0 to a bijection

$$\pi \colon \{1, \dots, n\} \to \{1, \dots, m\}$$

where

$$\frac{G_{\pi(i)}}{G_{\pi(i)-1}} \cong \frac{H_i}{H_{i-1}}$$

for all $1 \le i \le n$. Easy:

$$\pi(i) = \begin{cases} \pi_0(i) & 1 \le i \le n-1, \\ k & i = n. \end{cases}$$

3 Does the converse hold?

The factorization of a group G into simple subgroups is unique, much as the factorization of a natural number n into primes is unique. Furthermore, the prime factors of a natural number uniquely determine said number.

This gives rise to a natural question: do the composition factors of a group G uniquely determine the group? In other words, does the converse of Jordan-Holder hold? The answer is no:

$$1 \lhd Z_p \lhd Z_{p^2}$$

and

$$1 \lhd Z_p \lhd Z_p \times Z_p.$$

The composition factors are identical (the only non-trivial check is $Z_{p^2}/Z_p \cong Z_p \cong (Z_p \times Z_p)/Z_p$) yet Z_{p^2} and $Z_p \times Z_p$ are not isomorphic.